On the ‘Ordnungszahlen’ in Gentzen’s First Consistency Proof

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Introduction

[1] Gentzen, Die Widerspruchsfreiheit der reinen Zahlentheorie. Math. Ann. 112 (1936).

[2] Gentzen, Neue Fassung des Widerspruchsfreiheits-beweises für die reine Zahlentheorie. Forsch. zur Logik u. zur Grundl. d. ex. Wiss. Neue Folge 4 (1938)

[3] Kogan-Bernstein, Simplification of Gentzen's reductions in the classical arithmetic. Zap.Nauch.Sem. LOMI 105 (1981) English translation: J.MATH.SCI. Vol.22(3) (1983)

§1 Derivations [1]

§2 Gentzen's 'Ordnungszahlen' $(\mathcal{O}, <_{\mathrm{IR}})$ [1]

§3 Assignment of 'Ordnungszahlen' to derivations [1]

§4 Embedding of $(\mathcal{O}, <_{\mathrm{IR}})$ into $(On, <)$

§5 Transforming Gentzen's assignment
into the assignment in [3]

§6 Reduction steps on sequents [1]

§7 Reduction steps on derivations [1]

§8 Transition to multisuccedent seqents (LK)
yielding an "amalgamation" of [1] and [2]

§1 Derivations. Inductive Definition of $d \vdash \Pi$.

1. Atomic derivations (axioms)

2. $d_0 \vdash \Gamma \to F(a)$ & variable condition
   $\implies \mathrm{I}^{a}_{\forall x F} d_0 \vdash \Gamma \to \forall x F(x)$.

3. $d_0 \vdash \Gamma, A \to \bot \implies \mathrm{I}_{\neg A} d_0 \vdash \Gamma \to \neg A$.

4. $d_0 \vdash \Gamma \to F(0)$ & $d_1 \vdash F(a), \Delta \to F(\mathsf{S}a)$ & var.cond.
   $\implies \mathrm{Ind}^{a,t}_{F} d_0 d_1 \vdash \Gamma, \Delta \to F(t)$

5. If $d_i \vdash \Pi_i \;\; (i = 0, \ldots, l)$, and if $\dfrac{\Pi_0 \quad \ldots \quad \Pi_l}{\Pi}$ is a *chain inference of rank* r, then
   $\mathbf{K}^{r}_{\Pi} d_0 \ldots d_l \vdash \Pi$.

§2 Gentzen's 'Ordnungszahlen' $(\mathcal{O}, <_{\mathbb{R}})$

$u, v \in \{0,1\}^+$, $0^n := \overbrace{0\ldots0}^{n}$, $n.u \in \mathbb{R}$

Definition of $M_n \subseteq \{0,1\}^+$ (Mantissen)

1. $M_0 := \{1\}$;
2. $M_{n+1} := \{u_0 0^{n+1} u_1 0^{n+1} \ldots 0^{n+1} u_l :$
   $u_0, \ldots, u_l \in M_n \ \& \ 0.u_l \leq_{\mathbb{R}} \ldots \leq_{\mathbb{R}} 0.u_0\}$.

$M := \bigcup_{n \in \mathbb{N}} M_n$, $\mathsf{h}(u) := \mathbf{min}\{n : u \in M_n\}$

Remarks. (a) $M_n \subseteq M_{n+1}$.

(b) $\mathsf{h}(u)$ is the maximal number of consecutive zeros in u (*H̲öchstanzahl aufeinanderfolgender Nullen*).

Definition.

$\mathcal{O} := \{n.u : n \in \mathbb{N} \;\&\; u \in M_n\}$ (Ordnungszahlen)

Theorem 1. $\mathcal{O}$ is wellordered by $<_{\mathbb{R}}$.

§3 Assignment of 'Ordnungszahlen' to derivations

Definition of $\rho(d) \in \mathbb{N}$ (Numerus) and $\mu(d) \in M$ (Mantisse)

Case $d = \mathrm{K}^r_{\Pi} d_0 \ldots d_l$ (w.l.o.g. $l > 0$).

Let σ be a permut. s.t. $0.\mu(d_{\sigma(0)}) \geq_{\mathbb{R}} \ldots \geq_{\mathbb{R}} 0.\mu(d_{\sigma(l)})$.

$\mu(d) := u_0 0^{\nu+1} u_1 0^{\nu+1} \ldots 0^{\nu+1} u_l$,

where $u_i := \mu(d_{\sigma(i)})$ and $\nu := \mathbf{max}\{\mathrm{h}(u_0), \ldots, \mathrm{h}(u_l)\}$

Abbreviation. $\mathrm{h}(d) := \mathrm{h}(\mu(d))$, $\mathrm{h}'(d) := \mathrm{h}(d) - 1$.

Proposition. $\mathrm{h}'(d) = \mathbf{max}\{\mathrm{h}(d_0), \ldots, \mathrm{h}(d_l)\}$ $(***)$

The numerus $\rho(d)$ is defined by the equation
$\rho(d) - \mathsf{h}(d) = \mathbf{max}(\{(\rho(d_i) - \mathsf{h}(d_i)) - 1 : i \leq l\} \cup \{r\})$.

Remark.
With $\mathsf{dg}(d) := \rho(d) - \mathsf{h}(d)$, this equation becomes
$\mathsf{dg}(d) = \mathbf{max}(\{\mathsf{dg}(d_i) - 1 : i \leq l\} \cup \{r\})$.

Definition.
$\mathsf{Ord}(d) := \rho(d).\mu(d)$ (*Ordnungszahl* of d)

Since (by definition) $\mathsf{h}(\mu(d)) = \mathsf{h}(d) \leq \rho(d)$,

we have $\mu(d) \in M_{\rho(d)}$, i.e. $\underline{\mathsf{Ord}(d) \in \mathcal{O}}$.

Theorem 2. If d^- results from d by a *reduction step on derivations* then $\mathsf{Ord}(d^-) <_{\mathbb{R}} \mathsf{Ord}(d)$.

§4 Embedding of $(\mathcal{O}, <_{\mathbb{R}})$ into $(On, <)$

Definition of $|u|_n \in On$ for $u \in M_n$

1. $|1|_0 := 0$.
2. $|u_0 0^{n+1} \ldots 0^{n+1} u_l|_{n+1} := \omega^{|u_0|_n} + \ldots + \omega^{|u_l|_n}$.

Lemma 1. For $u \in M_n$ the following holds:

(a) $|u|_{n+k} = \omega_k(|u|_n)$,

(b) $\omega_n(0) \leq |u|_n < \omega_{n+1}(0)$.

Definition. For $n.u \in \mathcal{O}$ let $|n.u| := |u|_n$.

Lemma 2.

$n.u,\ m.v \in \mathcal{O}\ \&\ n.u <_{\mathbb{R}} m.v \ \Rightarrow\ |n.u| < |m.v|$.

Definition. $\mathsf{o}(\boldsymbol{d}) = |\mathsf{Ord}(\boldsymbol{d})|$.

Theorem 2 together with Lemma 2 yields: $\mathsf{o}(\boldsymbol{d}^{-}) < \mathsf{o}(\boldsymbol{d})$.

Our goal is, to find a direct recursive definition of $\mathsf{o}(\boldsymbol{d})$ which does not use the assignment $\mathsf{Ord}(d)$.

§5 Transforming Gentzen's assignment

Abbreviation. $\widetilde{\mathsf{o}}(\boldsymbol{d}) := |\boldsymbol{\mu}(\boldsymbol{d})|_{\mathsf{h}(\boldsymbol{d})}$

Lemma 3. $\mathsf{o}(\boldsymbol{d}) = \boldsymbol{\omega}_{\mathsf{dg}(\boldsymbol{d})}(\widetilde{\mathsf{o}}(\boldsymbol{d}))$.

Proof:

$$\mathsf{o}(\boldsymbol{d}) = |\boldsymbol{\rho}(\boldsymbol{d}).\boldsymbol{\mu}(\boldsymbol{d})| = |\boldsymbol{\mu}(\boldsymbol{d})|_{\boldsymbol{\rho}(\boldsymbol{d})} \overset{\text{L.1a}}{=} \boldsymbol{\omega}_{\boldsymbol{\rho}(\boldsymbol{d})-\mathsf{h}(\boldsymbol{d})}(|\boldsymbol{\mu}(\boldsymbol{d})|_{\mathsf{h}(\boldsymbol{d})}).$$

Lemma 4.

For $\boldsymbol{d} = \mathbf{K}^{r}_{\boldsymbol{\Pi}}\boldsymbol{d}_0 \dots \boldsymbol{d}_l$ we have $\widetilde{\mathsf{o}}(\boldsymbol{d}) = \boldsymbol{\omega}^{\boldsymbol{\alpha}_0} \# \dots \# \boldsymbol{\omega}^{\boldsymbol{\alpha}_l}$
with $\boldsymbol{\alpha}_i := \boldsymbol{\omega}_{\mathsf{h}'(\boldsymbol{d})-\mathsf{h}(\boldsymbol{d}_i)}(\widetilde{\mathsf{o}}(\boldsymbol{d}_i))$

Lemma 4.

For $\boldsymbol{d} = \mathbf{K}^r_{\boldsymbol{\Pi}} \boldsymbol{d}_0 \ldots \boldsymbol{d}_l$ we have $\widetilde{\mathsf{o}}(\boldsymbol{d}) = \boldsymbol{\omega}^{\boldsymbol{\alpha}_0} \# \ldots \# \boldsymbol{\omega}^{\boldsymbol{\alpha}_l}$ with $\boldsymbol{\alpha}_i := \boldsymbol{\omega}_{\mathsf{h}'(\boldsymbol{d}) - \mathsf{h}(\boldsymbol{d}_i)}(\widetilde{\mathsf{o}}(\boldsymbol{d}_i))$

Proof:

By definition, $\boldsymbol{\mu}(\boldsymbol{d}) = \boldsymbol{u}_0 0^{\boldsymbol{\nu}+1} \ldots 0^{\boldsymbol{\nu}+1} \boldsymbol{u}_l$ with $\boldsymbol{u}_{\boldsymbol{\sigma}'(\boldsymbol{i})} = \boldsymbol{\mu}(\boldsymbol{d}_i)$ and $\boldsymbol{\nu} = \mathbf{max}\{\mathsf{h}(\boldsymbol{u}_0), \ldots, \mathsf{h}(\boldsymbol{u}_l)\}$.

Hence $\boldsymbol{\nu} = \mathbf{max}\{\mathsf{h}(\boldsymbol{d}_0), \ldots, \mathsf{h}(\boldsymbol{d}_l)\} = \mathsf{h}'(\boldsymbol{d}) = \mathsf{h}(\boldsymbol{d}) - 1$,

$\widetilde{\mathsf{o}}(\boldsymbol{d}) \stackrel{\text{Def}}{=} |\boldsymbol{\mu}(\boldsymbol{d})|_{\mathsf{h}(\boldsymbol{d})} = |\boldsymbol{\mu}(\boldsymbol{d})|_{\boldsymbol{\nu}+1} = \boldsymbol{\omega}^{|\boldsymbol{u}_0|_{\boldsymbol{\nu}}} + \ldots + \boldsymbol{\omega}^{|\boldsymbol{u}_l|_{\boldsymbol{\nu}}}$,

$|\boldsymbol{u}_{\boldsymbol{\sigma}'(\boldsymbol{i})}|_{\boldsymbol{\nu}} = |\boldsymbol{\mu}(\boldsymbol{d}_i)|_{\boldsymbol{\nu}} \stackrel{\text{L.1a}}{=} \boldsymbol{\omega}_{\boldsymbol{\nu} - \mathsf{h}(\boldsymbol{d}_i)}(|\boldsymbol{\mu}(\boldsymbol{d}_i)|_{\mathsf{h}(\boldsymbol{d}_i)})$.

Summary. For $\boldsymbol{d} = \mathbf{K}^{r}_{\boldsymbol{\Pi}}\boldsymbol{d}_0 \ldots \boldsymbol{d}_l$ we have

- $\mathsf{dg}(\boldsymbol{d}) = \mathbf{max}(\{\mathsf{dg}(\boldsymbol{d_i}) - 1 : \boldsymbol{i} \leq \boldsymbol{l}\} \cup \{\boldsymbol{r}\}$
- $\mathsf{h}(\boldsymbol{d}) = \mathbf{max}\{\mathsf{h}(\boldsymbol{d}_0), \ldots, \mathsf{h}(\boldsymbol{d_l})\} + 1$
- $\widetilde{\mathsf{o}}(\boldsymbol{d}) = \boldsymbol{\omega}^{\alpha_0} \# \ldots \# \boldsymbol{\omega}^{\alpha_l}$ with $\boldsymbol{\alpha}_i := \boldsymbol{\omega}_{\mathsf{h}'(\boldsymbol{d}) - \mathsf{h}(\boldsymbol{d}_i)}(\widetilde{\mathsf{o}}(\boldsymbol{d_i}))$
- $\mathsf{o}(\boldsymbol{d}) = \boldsymbol{\omega}_{\mathsf{dg}(\boldsymbol{d})}(\widetilde{\mathsf{o}}(\boldsymbol{d}))$

Idea. Simplify this ordinal assignment by setting

$$\widetilde{\mathsf{o}}(\boldsymbol{d}) := \boldsymbol{\omega}^{\tilde{\mathsf{o}}(\boldsymbol{d}_0)} \# \ldots \# \boldsymbol{\omega}^{\tilde{\mathsf{o}}(\boldsymbol{d}_l)}$$

Then $\mathsf{h}(\boldsymbol{d})$ *becomes obsolete, and the proof of* $\mathsf{o}(\boldsymbol{d}^{-}) < \mathsf{o}(\boldsymbol{d})$ *works as well with this simpler* $\widetilde{\mathsf{o}}$*.*

Comparison with [3]: $\mathsf{h}(\boldsymbol{d}) =$ excess of $\boldsymbol{d}$, $\widetilde{\mathsf{o}}(d) = \mathsf{FO}(d)$

Preliminaries to part II (Gentzen 1936)

- Formulas $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{F}$ are build up from arithmetic prime formulas by $\forall$, $\wedge$, $\neg$.
  In the following, $\wedge$ is not mentioned.
- Sequents: $\boldsymbol{\Pi} = \boldsymbol{\Gamma} {\rightarrow} \boldsymbol{C}$,
  $\mathsf{L}(\boldsymbol{\Gamma} {\rightarrow} \boldsymbol{C}) := \boldsymbol{\Gamma}, \quad \mathsf{R}(\boldsymbol{\Gamma} {\rightarrow} \boldsymbol{C}) := \{\boldsymbol{C}\}$,
  $\boldsymbol{A}, \boldsymbol{\Pi} := \boldsymbol{A}, \boldsymbol{\Gamma} {\rightarrow} \boldsymbol{C} = (\{\boldsymbol{A}\} \cup \boldsymbol{\Gamma}) {\rightarrow} \boldsymbol{C}$,
  $\boldsymbol{\Pi}, \boldsymbol{A} := \boldsymbol{\Gamma} {\rightarrow} \boldsymbol{A}$ **!!!**
- $\boldsymbol{\Pi}$ has **endform** $:\Leftrightarrow \ \top \in \mathsf{R}(\boldsymbol{\Pi}) \vee \bot \in \mathsf{L}(\boldsymbol{\Pi})$.

§6 Reduction steps on sequents

$(\mathsf{R}_{\forall xF})$ $\dfrac{\ldots \Gamma \to F(n) \ldots (n \in \mathbb{N})}{\Gamma \to \forall x F(x)}$ **('Wahlfreiheit')**

$(\mathsf{L}^{k}_{\forall xF})$ $\dfrac{F(k), \Gamma \to C}{\Gamma \to C}$ if $\forall x F \in \Gamma$

$(\mathsf{R}_{\neg A})$ $\dfrac{A, \Gamma \to \bot}{\Gamma \to \neg A}$ $(\mathsf{L}^{0}_{\neg A})$ $\dfrac{\Gamma \to A}{\Gamma \to C}$ if $\neg A \in \Gamma$

$$\mathcal{I}(\Pi, n) := \begin{cases} \Pi, F(n) & \text{if } \mathcal{I} = \mathsf{R}_{\forall xF} \\ F(k), \Pi & \text{if } \mathcal{I} = \mathsf{L}^{k}_{\forall xF} \\ A, \Pi, \bot & \text{if } \mathcal{I} = \mathsf{R}_{\neg A} \\ \Pi, A & \text{if } \mathcal{I} = \mathsf{L}^{0}_{\neg A} \end{cases}$$

Definition of $\mathcal{I} \triangleright \Pi$ ($\mathcal{I}$ is permissible for Π)

$\mathsf{R}_B \triangleright \Pi \;:\Leftrightarrow\; B \in \mathsf{R}(\Pi)$

$\mathsf{L}^k_B \triangleright \Pi \;:\Leftrightarrow\; B \in \mathsf{L}(\Pi)$

Gentzen's Kettenschluß

$\dfrac{\Gamma_0 \to A_0 \quad \dots \quad \Gamma_l \to A_l}{\Gamma \to C}$ is a chain inference of degree r if

there exists a $j \le l$ such that $A_j = C$ and

$\forall i \le j (\Gamma_i \subseteq \Gamma, A_0, \dots, A_{i-1})$ and $\forall i < j (\mathrm{rk}(A_i) \le r)$.

§7 Reduction steps on derivations

For each derivation $\boldsymbol{d}$, whose endsequent is not in endform we shall now define *the reduction step on* $\boldsymbol{d}$ and at the same time **prove** the following:

by such a step the derivation $\boldsymbol{d}$ *is transformed into another derivation* $\boldsymbol{d}[\![\boldsymbol{n}]\!]$ *and its endsequent* $\boldsymbol{\Pi}$ *is thereby modified in the following way:*
At most one reduction step $\mathsf{tp}(\boldsymbol{d})$ *is carried out on the sequent. It may thus happen that an endsequent remains entirely unchanged.*
In this case we set $\mathsf{tp}(\boldsymbol{d}) := \mathsf{Rep}$ with $\mathsf{Rep}(\boldsymbol{\Pi}, \boldsymbol{n}) := \boldsymbol{\Pi}$.

Theorem 3. $\boldsymbol{d} \vdash \boldsymbol{\Pi} \;\Rightarrow\; \boldsymbol{d}[\![\boldsymbol{n}]\!] \vdash \mathsf{tp}(\boldsymbol{d})(\boldsymbol{\Pi}, \boldsymbol{n})$.

Inference symbols

$\mathsf{R}_{\boldsymbol{B}}$, $\mathsf{L}^{\boldsymbol{k}}_{\boldsymbol{B}}$ (reduction steps on sequents),
Rep, Ax_0.

syntactic variable for inference symbols: $\boldsymbol{\mathcal{I}}$

Definition.

$\mathsf{Ax}_0 \rhd \boldsymbol{\Pi} \quad :\Leftrightarrow \quad \boldsymbol{\Pi}$ has endform

$\mathsf{Rep} \rhd \boldsymbol{\Pi} \quad :\Leftrightarrow \quad 0 = 0$.

$\mathsf{Rep}(\boldsymbol{\Pi}, \boldsymbol{n}) := \boldsymbol{\Pi}$

Remark. $\boldsymbol{\mathcal{I}} \rhd \emptyset \to \bot \ \Rightarrow \ \boldsymbol{\mathcal{I}} = \mathsf{Rep}$.

Theorem 3. If $\boldsymbol{d} \vdash \boldsymbol{\Pi}$ then

(i) $\mathsf{tp}(\boldsymbol{d}) \triangleright \boldsymbol{\Pi}$;

(ii) $\mathsf{Ax}_0 \not\triangleright \boldsymbol{\Pi} \;\Rightarrow\; \boldsymbol{d}[\![\boldsymbol{n}]\!] \vdash \mathsf{tp}(\boldsymbol{d})(\boldsymbol{\Pi}, \boldsymbol{n}) \;\; (\forall \boldsymbol{n})$.

Corollary.

$\boldsymbol{d} \vdash \emptyset{\rightarrow}\bot \;\Rightarrow\; \boldsymbol{d}[\![0]\!] \vdash \emptyset{\rightarrow}\bot$.

Proof:

$\boldsymbol{d} \vdash \emptyset{\rightarrow}\bot \;\stackrel{\text{(i)+Remark}}{\Rightarrow}$

$\boldsymbol{d} \vdash \emptyset{\rightarrow}\bot \;\&\; \mathsf{tp}(\boldsymbol{d}) = \mathsf{Rep} \;\stackrel{\text{(ii)}}{\Rightarrow}$

$\boldsymbol{d}[\![0]\!] \vdash \mathsf{Rep}(\emptyset{\rightarrow}\bot, 0) = \emptyset{\rightarrow}\bot$.

Definition of $\mathsf{tp}(d)$ and $d[\![n]\!]$

Simultaneously one proves: $d \vdash \Pi \;\Rightarrow\; \mathsf{tp}(d) \triangleright \Pi$.

5. $d = \mathrm{K}^r_{\Pi} d_0 \ldots d_l \vdash \Pi$ with $d_i \vdash \Pi_i \;\; (i \leq l)$.
Let $j_0 := \mathbf{min}\{j \leq l : \mathsf{R}(\Pi_j) = \mathsf{R}(\Pi)\}$.

5.1. d **critical**, i.e. $\forall i \leq j_0 (\mathsf{tp}(d_i) \not\triangleright \Pi)$.
By IH $\forall i \leq j_0 (\mathsf{tp}(d_i) \triangleright \Pi_i)$. It follows that there exist $i < j \leq j_0$ and B, k s.t. $\mathsf{tp}(d_i) = \mathsf{R}_B$ & $\mathsf{tp}(d_j) = \mathsf{L}^k_B$.

$\mathsf{tp}(d) := \mathsf{Rep}$ and $d[\![0]\!] := \mathrm{K}^{r-1}_{\Pi} d\{0\} d\{1\}$ with

$$d\{0\} := \mathrm{K}^r_{\mathsf{tp}(d_i)(\Pi,k)} d_0 \ldots d_{i-1} d_i[\![k]\!] d_{i+1} \ldots d_l,$$

$$d\{1\} := \mathrm{K}^r_{\mathsf{tp}(d_j)(\Pi,0)} d_0 \ldots d_{j-1} d_j[\![0]\!] d_{j+1} \ldots d_l.$$

5.2. d **not critical**:

Let i be minimal such that $\mathrm{tp}(d_i) \triangleright \Pi$.

5.2.1. d_i critical: $\mathrm{tp}(d) := \mathrm{Rep}$ and

$d[\![0]\!] := \mathbf{K}^{r'}_{\Pi} d_0 \dots d_{i-1} d_i\{0\} d_i\{1\} d_{i+1} \dots d_l$

5.2.2. d_i not critical:

$\mathrm{tp}(d) := \mathrm{tp}(d_i)$,

$d[\![n]\!] := \mathbf{K}^{r}_{\mathrm{tp}(d_i)(\Pi,n)} d_0 \dots d_{i-1} d_i[\![n]\!] d_{i+1} \dots d_l$.

§8 Transition to multisuccedent seqents (LK)

$\Pi = \Gamma \to \Delta$,
L(Π) := Γ, R(Π) := Δ.
$A, \Pi := A, \Gamma \to \Delta := (\{A\} \cup \Gamma) \to \Delta$ (as before)
$\Pi, A := \Gamma \to \Delta, A := \Gamma \to (\Delta \cup \{A\})$

Kettenschluß: $\dfrac{\Pi_0 \quad \ldots \quad \Pi_l}{\Pi}$ is a chain inference of degree r if Π can be derived from $\Pi_0, \ldots, \Pi_l$ by a finite number of cuts of degree $\leq r$.

Definition of $\mathrm{tp}(d)$ and $d[\![n]\!]$

5. $d = \mathrm{K}^r_{\Pi} d_0 \ldots d_l \vdash \Pi$ with $d_i \vdash \Pi_i$ $(i \le l)$.

Let $j_0 := l$. Then literally as before.

Lemma (*Existence of a suitable cut*)

If $\dfrac{\Pi_0 \quad \ldots \quad \Pi_l}{\Pi}$ is a chain inference of degree r, and if $\forall i \le l (\mathcal{I}_i \triangleright \Pi_i \;\&\; \mathcal{I}_i \not\triangleright \Pi)$ then

$\exists i, j \le l \exists k, B (\mathcal{I}_i = \mathsf{R}_B \;\&\; \mathcal{I}_j = \mathsf{L}^k_B \;\&\; \mathrm{rk}(B) \le r)$.