Transitivity Elimination: Where and Why

Pierluigi Minari

Section of Philosophy, DILEF, University of Florence
minari@unifi.it

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Intuitive motivations: two analogies

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Analogy 1: Modus ponens / transitivity rule for equality:

$$\frac{A \to B \qquad A}{B} \qquad \approx \qquad \frac{t = r \qquad r = s}{t = s}$$

- Standard presentation of an equational proof system $\mathbf{E}$:
  - *certain specific axioms* (a given set E of equation **schemas**)
  - the usual inference rules for equality:

$$\frac{}{t = t}\,[\textit{refl}] \qquad \frac{t = s}{s = t}\,[\textit{symm}] \qquad \frac{t = r \quad r = s}{t = s}\,[\textit{trs}]$$

$$\frac{t_i = s_i}{f^n(t_1, \ldots, t_n) = f^n(t_1 \ldots, t_{i-1}, s_i, t_{i+1}, \ldots, t_n)}\,[\textit{congr: } 1 \leq i \leq n]$$

- Birkhoff's completeness theorem for equational logic (1935):

$$\vdash_{\mathbf{E}} t = s \qquad \Leftrightarrow \qquad E \models t = s$$

- The **transitivity rule** (which cannot be dispensed with, except that in trivial cases) **has an inherently *synthetic* character** in combining derivations, like *modus ponens* in Hilbert-style proof systems
- Potential loss of relevant information along formal derivations (**no kind of "subterm property" available!**)
- As a consequence, *naive* proof-theoretic arguments are usually inapplicable (e.g.: syntactic consistency proofs by induction on the length of derivations)
- In general, derivations lack any significant mathematical structure
- All in all, 'synthetic' equational calculi do not lend themselves *directly* to proof-theoretical analysis

Question

Are there significant cases in which it is both *possible* and *useful* to turn a ‘synthetic’ equational proof system into an **equivalent** ‘analytic’ proof system, one in which **the transitivity rule is provably *redundant***?

Analogy 2: Cut-elimination / Church-Rosser:

(syntactic) proofs of *cut-elimination* for Gentzen-style calculi

$$\approx^{?}$$

some proofs of the *Church-Rosser* theorem for (weak, β-, $\beta\eta$-) reduction (e.g., proofs *à la* Tait & Martin-Löf, using parallel reduction)

(Partial) answer: *where*

Equational theories of type-free operations, in particular:

combinatory logic (more generally: arbitrary "combinatory systems")

and **lambda-calculus**

can be presented through an 'analytic' proof system.

Answer: *why*

- *Conceptual* interest: **analyticity** at work in an equational environment (*subterm property*); *direct* consistency proofs.
- New (and short) proofs of well-known key results concerning reductions (like *Confluence*, *Standardization*, *Leftmost reduction*, η-*Postponement*) can be given in a unified framework by purely proof-theoretical methods.
- Decidability of (pure, linear and recursive) fragments of CL with *extensionality*, like e.g. $\mathbf{BCK} +$ *ext*.
- Positive solution of Curry's problem (1958) on *combinatory strong reduction* (a "*Methodenreinheit*" issue).
- ...

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"normalizability" of transitivity-free derivations

$\Downarrow$

applications to combinatory / lambda reductions

- **Combinatory Logic**: CL (& generalizations)

  P.M. 2004, *Analytic combinatory calculi and the elimination of transitivity*, Arch. Math. Logic 43, 159-191.

- **Lambda-Calculus**: $\lambda\beta$, $\lambda\beta\eta$

  P.M. 2005, *Proof-theoretical methods in combinatory logic and λ-calculus*, in: S. Cooper et al. (Eds), **CiE 2005: New Computational Paradigms**, Amsterdam, 148-157.

  P.M. 2007, *Analytic proof systems for λ-calculus: the elimination of transitivity, and why it matters*, Arch. Math. Logic 46, 385-424.

- **Extensional Combinatory Logic**: $\mathrm{CL}_{\mathbf{ext}}$ (& generalizations)

  P.M. 2009, *A solution to Curry and Hindley's problem on combinatory strong reduction*, Arch. Math. Logic 48, 159-184.

Synthetic systems

CL

- Axiom schemas:

$$\mathsf{K}ts = t \qquad \mathsf{S}tsr = tr(sr) \qquad [\ \mathsf{I}t = t\ ]$$

- Inference rules:

ϱ *(reflexivity)* $\quad \sigma$ *(symmetry)* $\quad \tau$ *(transitivity)*

$$\frac{t = s}{rt = rs}\,\mu \qquad \frac{t = s}{tr = sr}\,\nu$$

(app-congruence rules)

$\mathbf{CL_{ext}}$

$$:= \ \mathbf{CL} + \ \frac{tx = sx}{t = s}\,Ext \quad \{x \notin V(ts)\}$$

CL generalized

A **combinatory system** $\mathbb{X}$ is a map, defined on a non-empty set $\mathbf{X} = dom(\mathbb{X})$ of primitive combinators $(\mathsf{F}, \mathsf{G} \ldots)$, which associates to each $\mathsf{F} \in \mathbf{X}$ a pair $\langle k_\mathsf{F}, d_\mathsf{F} \rangle$ s.t.:

- k_F, the *index* of F under $\mathbb{X}$, is a non negative integer;
- d_F, the *definition* of F under $\mathbb{X}$, is a term with $V(d_\mathsf{F}) \subseteq \{v_1, \ldots, v_{k_\mathsf{F}}\}$.

Intuitively, $\mathbb{X}$ fixes an axiom schema for each primitive combinator $\mathsf{F} \in \mathbf{X}$:

$$\mathsf{F} t_1 \ldots t_{k_\mathsf{F}} = d_\mathsf{F}[v_1/t_1, \ldots, v_{k_\mathsf{F}}/t_{k_\mathsf{F}}] \qquad (\mathrm{AX}\,\mathsf{F})_\mathbb{X}$$

$\mathbf{CL}[\mathbb{X}]$ / $\mathbf{CL}_{\mathbf{ext}}[\mathbb{X}]$ are now defined exactly as $\mathbf{CL}$ / $\mathbf{CL}_{\mathbf{ext}}$, except that the axiom schemas for the combinators K, S (I) are replaced by the set

$$\{(\mathrm{AX}\,\mathsf{F})_\mathbb{X} \mid \mathsf{F} \in \mathbf{X}\}$$

of axiom schemas induced by $\mathbb{X}$.

Example 1

The familiar pair $\{\mathsf{K}, \mathsf{S}\}$ of primitive combinators of $\mathbf{CL}$ corresponds to the combinatory system $\mathbb{C}$ such that:

- $\mathbf{C} = \{\mathsf{K}, \mathsf{S}\}$
- $k_{\mathsf{K}} = 2$ and $d_{\mathsf{K}} = v_1$
- $k_{\mathsf{S}} = 3$ and $d_{\mathsf{S}} = v_1 v_3 (v_2 v_3)$

Example 2

But also the following is a perfectly legitimate (maybe odd!) combinatory system:

- $\mathbf{X} = \{\mathsf{F}, \mathsf{G}, \mathsf{H}\}$
- $k_{\mathsf{F}} = 0$ and $d_{\mathsf{F}} = \mathsf{FF}(\mathsf{HG})(\mathsf{GH})$
- $k_{\mathsf{G}} = 2$ and $d_{\mathsf{G}} = v_2 \mathsf{GH} v_1$
- $k_{\mathsf{H}} = 4$ and $d_{\mathsf{H}} = v_1 v_1 ((\mathsf{H} v_2)(\mathsf{F} v_1))(v_4 \mathsf{H})$

$\boldsymbol{\lambda\beta}$

- Axiom schema:

$$(\lambda x.t)r = t[x/r] \qquad \textit{(β-conversion)}$$

- Inference rules: ϱ, σ, τ, μ, ν, plus

$$\frac{t = s}{\lambda x.t = \lambda x.s}\,\xi \qquad \textit{(abstr-congruence rule)}$$

$\boldsymbol{\lambda\beta\eta}$

$:= \ \boldsymbol{\lambda\beta} + \quad \lambda x.tx = t \ \{x \notin V(t)\}$ *(η-conversion)*

or, equivalently, $\boldsymbol{\lambda\beta} + \mathit{Ext}$

Analytic "A"-systems: main features

1. **combinatory axiom schemas / β-conversion schema** are replaced by **introduction rules** (to the left / to the right), as follows.

Example: introduction rules for the combinator S

$$\mathsf{S}tsr = tr(sr) \quad [\mathrm{AX}\,\mathsf{S}]$$

$$\Downarrow\Downarrow$$

$$\frac{tr(sr)p_1 \ldots p_n = q}{\mathsf{S}tsrp_1 \ldots p_n = q}\,[\mathsf{S}_l] \qquad \frac{q = tr(sr)p_1 \ldots p_n}{q = \mathsf{S}tsrp_1 \ldots p_n}\,[\mathsf{S}_r]$$

where $n \geq 0$, i.e.: the "side terms" $p_1, \ldots, p_n$ may be missing

In general: introduction rules for a primitive combinator F

$$\mathsf{F}t_1 \dots t_{k_\mathsf{F}} = d_\mathsf{F}[t_1, \dots, t_{k_\mathsf{F}}] \quad (\mathrm{AX}\,\mathsf{F})_\mathbb{X}$$

$$\Downarrow\Downarrow$$

$$\frac{d_\mathsf{F}[t_1, \dots, t_{k_\mathsf{F}}]p_1 \dots p_n = s}{\mathsf{F}t_1 \dots t_{k_\mathsf{F}} p_1 \dots p_n = s}\,[\mathsf{F}_l]_\mathbb{X} \qquad \frac{s = d_\mathsf{F}[t_1, \dots, t_{k_\mathsf{F}}]p_1 \dots p_n}{s = \mathsf{F}t_1 \dots t_{k_\mathsf{F}} p_1 \dots p_n}\,[\mathsf{F}_r]_\mathbb{X}$$

where $n \geq 0$.

(We write $t[s_1, \dots, s_n]$ short for $t[v_1/s_1, \dots, v_n/s_n]$)

β-introduction rules

$$(\lambda x.t)r = t[x/r] \quad [\beta\text{-conv}]$$

$$\Downarrow \Downarrow$$

$$\frac{t[x/r]p_1 \ldots p_n = q}{(\lambda x.t)rp_1 \ldots p_n = q}\,[\beta_l] \qquad \frac{q = t[x/r]p_1 \ldots p_n}{q = (\lambda x.t)rp_1 \ldots p_n}\,[\beta_r]$$

where $n \geq 0$, i.e.: the "side terms" $p_1, \ldots, p_n$ may be missing

Analytic "A"-systems: main features (contd)

1. **combinatory axiom schemas / β-conversion schema** are turned into the corresponding **left/right introduction rules**, as shown.
2. **symmetry rule** ▶ dropped
3. **reflexivity (0-premises) rule** ▶ restricted to atomic terms (ϱ')
4. ***app*-congruence (= monotony) rule(s)**
   - ▶ taken in the **parallel** version

$$\frac{t = s \quad p = q}{tp = sq}\,App$$

5. **extensionality (if any)**
   - ▶ **always** taken in the rule-version

$$\frac{tx = sx}{t = s}\,Ext \qquad \{x \notin V(ts)\}$$

Analytic "A"-systems (summarizing)

$$\mathbf{A}_{\ldots}[\ldots] \;:=\; \begin{cases} \varrho',\ App,\ \tau\ /\ \text{and}\ \xi & \textit{“structural”}\ \text{rules} \\ \mathsf{F}_l,\ \mathsf{F}_r\ /\ \text{or}\ \beta_l,\ \beta_r & \textit{introduction}\ \text{rules} \\ \big(Ext & \textit{extensionality}\ \text{rule}\big) \end{cases}$$

Synthetic systems	$\rightsquigarrow$	Equivalent analytic systems
$\mathbf{CL}$	$\rightsquigarrow$	$\mathbf{A}[\mathbb{C}]$
$\mathbf{CL}[\mathbb{X}]$	$\rightsquigarrow$	$\mathbf{A}[\mathbb{X}]$
$\mathbf{CL}_{\mathbf{ext}}$ / $\mathbf{CL}_{\mathbf{ext}}[\mathbb{X}]$	$\rightsquigarrow$	$\mathbf{A}_{\mathbf{ext}}[\mathbb{C}]$ / $\mathbf{A}_{\mathbf{ext}}[\mathbb{X}]$
$\lambda\beta$	$\rightsquigarrow$	$\mathbf{A}[\beta]$
$\lambda\beta\eta$	$\rightsquigarrow$	$\mathbf{A}_{\mathbf{ext}}[\beta]$

Main Result

τ-elimination

A-systems admit (effective) transitivity elimination

Proofs (listed in order of increasing complexity):

- $\mathbf{A}[\mathbb{X}]$ ($\mathbb{X}$ arbitrary)
- $\mathbf{A_{ext}}[\mathbb{X}]$ **($\mathbb{X}$ linear)**
- $\mathbf{A}[\beta]$ and $\mathbf{A_{ext}}[\beta]$
- $\mathbf{A_{ext}}[\mathbb{X}]$ **($\mathbb{X}$ arbitrary)**

Immediate applications of τ-elimination

Fact

Transitivity-free derivations $\mathcal{D} \vdash t = s$ trivially enjoy a sort of **subterm property**. Namely:

- for A-Systems without Ext:
  if $\mathcal{D}$ contains an application of a left (right) F-intro, resp. of a left (right) β-intro, then t (s) contains a F-redex, resp. a β-redex.
- for A-systems with Ext:
  if $\mathcal{D}$ contains an application of a left (right) F-intro, resp. of a left (right) β-intro, then t (s) contains an occurrence of F, resp. an occurrence of λ.

Corollary 1 [Consistency]

For every analytic system $\mathbf{A}$:

- $\mathbf{A} \nvdash x = y$ (with x distinct from y)
- $\mathbf{A}$ (hence the corresponding synthetic system) is **consistent**

Corollary 2 [Church-Rosser]

The reductions

- $\twoheadrightarrow_{\mathbb{X}}$ (*weak* $\mathbb{X}$-combinatory reduction)
- $\twoheadrightarrow_{\beta}$, $\twoheadrightarrow_{\beta\eta}$ (β- and $\beta\eta$-reduction)
- $\succ\!\!-$ (*strong* combinatory reduction)

are confluent

Proof.
From any given τ-free derivation $\mathcal{D} \vdash t = s$ in an analytic system $\mathbf{A}$, we can extract a term $\boldsymbol{r}$ such that

$$t \twoheadrightarrow \boldsymbol{r} \twoheadleftarrow s$$

by straightforward induction on the length of $\mathcal{D}$
(where $\twoheadrightarrow$ is the appropriate reduction, depending on $\mathbf{A}$).

Further applications

Normal form(s) of τ-free derivations

τ-free derivations have nice structural properties and can be shown to normalize to suitable normal forms.

By exploiting this feature, new **very short** demonstrations of well known results concerning **reductions** can be given, including:

- Standardization
- Leftmost reduction (in particular for $\lambda\beta\eta$-reduction)
- η-*Postponement*, ...

On can also prove, e.g.

- the decidability of a natural class of fragments of $\mathbf{CL}_{\mathbf{ext}}$

τ-elimination

$\mathbf{A}[\mathbb{X}]$ systems — Proof strategy

We show how to eliminate a topmost application of τ :

$$\mathcal{D}_1 \vdash^- t = s\,,\;\; \mathcal{D}_2 \vdash^- s = r \qquad \blacktriangleright \qquad \mathcal{D}^* \vdash^- t = r$$

The *proof* runs by ω^3-induction:

main: $\mathrm{h}'(\mathcal{D}_1) + \mathrm{h}'(\mathcal{D}_2)$

secondary: $\mathrm{s}(\mathcal{D}_1) + \mathrm{s}(\mathcal{D}_2)$

ternary: $\|s\|$

This strategy doesn't work when the **extensionality rule** is present, coupled with **non linear** combinators.

$\mathbf{A}_{\mathbf{ext}}[\mathbb{X}]$ systems — Proof strategy

We show that the following **deep transitivity rule**

$$\frac{t = s \qquad \Phi[\![s]\!] = r}{\Phi[\![t]\!] = r}\ \tau^*$$

is eliminable.

The proof consists of **four** main steps (in this order):

1. deep F-inversion
2. left τ-elimination
3. deep F-introduction
4. elimination of a topmost occurrence of $[\tau^*]$

Step 1: *deep* F*-inversion* Lemma

The following **deep** combinatory *inversion* rules are τ-free admissible for any $\mathsf{F} \in \mathbf{X}$:

$$\frac{\Phi[\![\mathsf{F}t_1 \ldots t_k]\!] = s}{\Phi[\![d_{\mathsf{F}}[t_1, \ldots, t_k]]\!] = s}\,[\mathsf{F}_l^{\mathsf{inv}}] \qquad \frac{s = \Phi[\![\mathsf{F}t_1 \ldots t_k]\!]}{s = \Phi[\![d_{\mathsf{F}}[t_1, \ldots, t_k]]\!]}\,[\mathsf{F}_r^{\mathsf{inv}}]$$

Moreover, $[\mathsf{F}_l^{\mathsf{inv}}]$ and $[\mathsf{F}_r^{\mathsf{inv}}]$ preserve *right-handedness*, resp. *left-handedness*.

Proof.

"Marking" technique . . . □

Step 2: **left** τ*-elimination*

To any given pair

$$\mathcal{D}_1 \vdash_L^- t = s \qquad \textit{and} \qquad \mathcal{D}_2 \vdash^- s = r$$

of τ-free derivations, such that $\mathcal{D}_1$ is a left derivation, we can effectively associate a τ-free derivation

$$\mathcal{D}^* \vdash^- t = r$$

which is a left derivation provided $\mathcal{D}_2$ is such.

Proof.

Main induction on $\mathrm{s}(\mathcal{D}_2)$, secondary induction on $\mathrm{s}(\mathcal{D}_1)$, ternary induction on $\|s\|$, using deep F-inversion. □

Step 3: *deep* F*-introduction*

The following **deep** combinatory *introduction* rules are τ-free admissible for any $\mathsf{F} \in \mathbf{X}$:

$$\frac{\Phi[\![d_{\mathsf{F}}[t_1, \ldots, t_k]]\!] = s}{\Phi[\![\mathsf{F} t_1 \ldots t_k]\!] = s}\ [\mathsf{F}_l^+] \qquad \frac{s = \Phi[\![d_{\mathsf{F}}[t_1, \ldots, t_k]]\!]}{s = \Phi[\![\mathsf{F} t_1 \ldots t_k]\!]}\ [\mathsf{F}_r^+]$$

Moreover, $[\mathsf{F}_l^+]$ and $[\mathsf{F}_r^+]$ preserve *left-handedness*, resp. *right-handedness*.

Proof.

By left τ-elimination. □

Final step: *main elimination* Lemma

To each pair of τ-free derivations

$$\mathcal{D}_1 \vdash^- t = s \quad \textit{and} \quad \mathcal{D}_2 \vdash^- \Phi[\![s]\!] = r$$

we can effectively associate a τ-free derivation

$$\mathcal{D}^* \vdash^- \Phi[\![t]\!] = r$$

Proof.

We use deep F introduction and inversion.
The proof runs by ω^3-induction

- main: $\mathrm{s}(\mathcal{D}_1)$
- secondary: $\|s\|$
- ternary: $\mathrm{h}(\mathcal{D}_2)$

taking main cases according to the last inference R of $\mathcal{D}_1$. □

Combinatory strong reduction

Primitive combinators: I , K , S

$$\overline{t \succ t}\ \rho \qquad \overline{\mathsf{I}t \succ t}\ \mathsf{I} \qquad \overline{\mathsf{K}ts \succ t}\ \mathsf{K} \qquad \overline{\mathsf{S}tsr \succ tr(sr)}\ \mathsf{S}$$

$$\frac{t \succ s}{rt \succ rs}\ \mu \qquad \frac{t \succ s}{tr \succ sr}\ \nu \qquad \frac{t \succ r \quad r \succ s}{t \succ s}\ \tau$$

$$\frac{t \succ s}{\lambda^* x.t \succ \lambda^* x.s}\ \xi$$

Abstraction is defined according to the *strong* algorithm.

Curry's problem concerning $\succ$

H. B. Curry and R. Feys, Combinatory Logic, Vol. I, 1958

List of "Unsolved problems" in § 6 F.5

> *"c. Is it possible to prove the Church-Rosser property directly for strong reduction, without having recourse to transformations between that theory and the theory of λ-conversion? . . . "*

Remark

A solution was advanced by K. Loewen in 1968.
His proof, however, doesn't work because of a serious mistake — as pointed out in Hindley's MR review (1970).

Hindley's statement of the problem (2006)

— Problem #1, TLCA List of Open Problems, http://tlca.di.unito.it/opltlca/

Submitted by Roger Hindley *Date:* Known since 1958!

Statement. Is there a direct proof of the confluence of $\beta\eta$-strong reduction?

Problem Origin. First posed by Haskell Curry and Roger Hindley.

The $\beta\eta$-strong reduction is the combinatory analogue of $\beta\eta$-reduction in λ-calculus. It is confluent. Its only known confluence-proof is very easy, [Curry and Feys, 1958, 6F, p. 221 Theorem 3], but it depends on the having already proved the confluence of $\lambda\beta\eta$-reduction. Thus the theory of combinators is not self-contained at present. **Is there a confluence proof independent of λ-calculus?**

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Our confluence proof for $\succ$ is independent of λ-calculus!