From second order Analysis to subsystems of set theory

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The ordinal analysis of an axiom system T is the computation of the order–type of the shortest well–ordering that is elementarily definable in the language of T but whose well–foundedness is not provable by the means of T.

Equivalently the proof theoretic ordinal $|T|$ of a theory T is the supremum of order–types of elementarily definable well–orderings the well–foundedness of which is provable in T.

Theorem (ω–completeness)

A Π^1_1–sentence $(\forall X)F(X)$ is true in the standard structure $\mathbb{N}$ if there is a cut–free proof of $F(X)$ in first order ω–logic.

Definition (Truth complexity)

The truth complexity $\mathrm{tc}\big((\forall X)F(X)\big)$ of a Π^1_1–sentence $(\forall X)F(X)$ is the minimum of the order–types of cut free proof trees for $F(X)$ in first order ω–logic.

Theorem (Boundedness)

Let $\prec$ be a well–ordering on the natural numbers of limit order type. Then the order–type of $\prec$ is equal to the truth complexity of the sentence
$(\forall X)[(\forall x)\big[(\forall y{\prec}x)y \in X \to x \in X] \;\to\; (\forall x)[x \in X]\big]$.

Hence

$$|T| \leq \sup \{\mathrm{tc}\big(F\big) \mid\; T \vdash F\}$$

where $|T|$ means the proof theoretic ordinal of the axiom system T.

The standard procedure for ordinal analysis of predicative theories.

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$$T \vdash F \;\Rightarrow\; \exists \alpha, \rho \;\; \vdash^{\alpha}_{\rho} F$$

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$$T \vdash F \;\Rightarrow\; \exists \alpha, \rho \;\; \Big|\frac{\alpha}{\rho}\, F \;\Rightarrow\; \exists f \;\; \Big|\frac{f(\alpha,\rho)}{0}\, F\,.$$

Hence $|T| \leq \min\{\pi \,|\; \alpha < \pi \wedge \rho < \pi \wedge \pi \text{ is closed under } f\}$.

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The standard procedure for ordinal analysis of predicative theories.

$$T \vdash F \;\Rightarrow\; \exists \alpha, \rho \; \left|\frac{\alpha}{\rho}\right. F \;\Rightarrow\; \exists f \; \left|\frac{f(\alpha,\rho)}{0}\right. F\,.$$

Hence $|T| \leq \min\{\pi |\; \alpha < \pi \wedge \rho < \pi \wedge \pi$ is closed under $f\}$.

For predicative systems the function f is essentially the Veblen function. More precisely we have $f(\alpha, \rho) < \varphi_\sigma(0)$ for $\rho < \omega^\sigma$ and $\alpha < \varphi_\sigma(0)$.

Ramified Analysis is a ramified second order language extending first order arithmetic by ramified comprehension.

$(\mathrm{CA})^\alpha$ If $F(x)$ is a formula of stage less than α then $\{x \mid F(x)\}$ is a second order term of stage α. The stage of a formula F is the maximum of the stages of second order terms occurring in F.

Analogous to ω–logic there are infinitary rules

$(\bigwedge)$ $\vdash_{\rho}^{\alpha_\xi} \Delta, F(T)$ and $\alpha_\xi < \alpha$ for all terms T of stages $\xi < \eta$ implies $\vdash_{\rho}^{\alpha} \Delta, (\forall X^\eta) F(X^\eta)$ and dually

$(\bigvee)$ $\vdash_{\rho}^{\alpha_0} \Delta, F(T)$ and $\alpha_0 < \alpha$ for some T of stage $\xi < \eta$ implies $\vdash_{\rho}^{\alpha} \Delta, (\exists X^\eta) F(X^\eta)$.

This yields the Schütte–Feferman ordinal $\Gamma_0 := \min\{\alpha \mid \varphi_\alpha(0) = \alpha\}$ — the least ordinal that is closed under φ viewed as a binary function — as an upper bound for the order–type of infinite well–orders that are autonomously reachable.

Γ_0 is thus regarded as the bounding ordinal for predicativity.

Let $\prec$ be a given well–ordering of order–type ν. The system for ν–fold iterated inductive definitions ID_ν extends the language of Peano arithmetic by a constant $I_{F,\sigma}$ for every σ in the field of $\prec$ and every X–positive arithmetical formula $F(X,Y,x)$. Besides the axioms for Peano arithmetic we have the defining axioms for the constants $I_{F,\sigma}$

$$(\forall x)[F(I_{F,\sigma}, I_{\prec\sigma}, x) \to x \in I_{F,\sigma}]$$

and

$$(\forall x)[F(G, I_{\prec\sigma}, x) \to G(x)] \to I_{F,\sigma} \subseteq \{x|\ G(x)\},$$

where

$$I_{\prec\sigma} := \bigcup \{I_{G,\rho}|\ \rho \prec \sigma \wedge G(X,Y,x) \text{ is } X\text{–positive}\}.$$

This is a passage of his 1927 talk given in Hamburg

"Der Physiker verlangt gerade von einer Theorie, daß ohne die Heranziehung von anderweitiger Bedingungen aus den Naturgesetzen oder Hypothesen die besonderen Sätze allein durch Schlüsse, also auf Grund eines reinen Formelspiels abgeleitet werden. Nur gewisse Kombinationen und Folgerungen der physikalischen Gesetze können durch Experimente kontrolliert werden — so wie in meiner Beweistheorie nur die realen Aussagen unmittelbar einer Verifikation fähig sind."

Which in my translation says:

The physicist requires for a theory that its theorems can be formally derived from the laws of nature and its hypotheses alone without referring to outside perceptions. Only certain combinations and conclusions of physical laws are checkable by experiments — this also true for my proof theory in which only "real statements" are verifiable.

In analogy to the situation in physics we define a design of an "experiment" that checks the "real statements" of a mathematical theory T.

Definition

An experiment–design for an axiom system T is a function F_T such that for any Π^0_2–statement $(\forall x)(\exists y)R(x,y)$ and all $m \in \mathbb{N}$ the sentence $(\exists y)R(m,y)$ is a theorem of T iff there is an $n < F_T(m)$ such that $R(m,n)$.

Definition (Π^0_2–analysis of an axiom system T)

Extract by elimination of the "ideal means" in T–proofs of Π^0_2–statements a computable function F_T which designs an experiment for T. Clearly F_T should be obtainable without reference to ideal means.

$$n \in I_F \;\Leftrightarrow\; (\forall X)\big[(\forall y)[F(X,y) \to y \in X] \;\to\; n \in X\big]$$

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The stages I_F^α of a positive inductive definition are given by

$$n \in I_F^\alpha \;\; :\Leftrightarrow \;\; F(I_F^{<\alpha}, n) \qquad |n|_F = \min\{\alpha \mid \; n \in I_F^\alpha\}$$

where $I_F^{<\alpha} = \bigcup_{\xi<\alpha} I_F^\xi$.

Then there is a least ordinal $|F|$, the closure ordinal of F, such that

$$I_F = I_F^{|F|}.$$

For a well–ordering $\prec$ let
$P(X,x) \;:\Leftrightarrow\; (\forall y)[y \prec x \to y \in X] \;\to\; x \in X.$
Then

$$\mathrm{otyp}_{\prec}(n) = |n|_P = \mathrm{tc}\Big((\forall X)\big[(\forall x)P(X,x) \to n \in X\big]\Big)$$

and

$$\mathrm{otyp}(\prec) = |P| = \mathrm{tc}\Big((\forall X)\big[(\forall x)P(X,x) \to (\forall x)[x \in X]\big]\Big).$$

Hence

$$|T| = \sup\{|n|_F \mid F(X,x) \text{ is arithmetical, } X\text{–positive} \wedge T \vdash n \in I_F.\}$$

Defining

- $n \in I_F^\xi \;\; :\Leftrightarrow \;\; F(I_F^{<\xi}, n)$ and
  $n \in I_F^{<\alpha} \;\; :\Leftrightarrow \;\; \bigvee\{n \in I_F^\xi \mid \xi < \alpha\}$

leads to the rules

($\bigvee$) $\left|\frac{\alpha_0}{\rho}\right. \Delta, n \in I_F^\xi$ for some $\xi < \eta$ implies $\left|\frac{\alpha}{\rho}\right. \Delta, n \in I_F^{<\eta}$ for all $\alpha > \alpha_0$

($\bigwedge$) $\left|\frac{\alpha_\xi}{\rho}\right. \Delta, n \notin I_F^\xi$ and $\alpha_\xi < \alpha$ for all $\xi < \eta$ implies $\left|\frac{\alpha}{\rho}\right. \Delta, n \notin I_F^{<\eta}$

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The rules that axiomatize the initial ordinals Ω_κ (e.g., $\Omega_1 = \omega_1^{CK}, \ldots$) are

(Ω_κ) $\left|\frac{\alpha_0}{\rho}\right. \Delta\, , n \in I_{F,\kappa}^{\Omega_\kappa}$ implies $\left|\frac{\alpha}{\rho}\right. \Delta\, , n \in I_{F,\kappa}^{<\Omega_\kappa}$ for all $\alpha > \alpha_0$.

Boundedness Property.

- $\vdash^{\alpha}_{\kappa} \Delta, n \in I^{\xi}_{F,\kappa}$ implies $\vdash^{\alpha}_{0} \Delta, n \in I^{\eta}_{F,\kappa}$ for all $\eta \geq \alpha$.

  Hence $|n|_F \leq \alpha$ for arithmetical $F(X,x)$.

Lemma

From $\left|\frac{\alpha}{\rho}\right. \Delta, \Gamma$ *where* Γ *is a finite set of false sentences we obtain* $\left|\frac{\alpha}{0}\right. \Delta$.

The proof by induction on α is obvious. In case of a cut $\left|\frac{\alpha_0}{\rho}\right. \Delta, \Gamma, F$ and $\left|\frac{\alpha_0}{\rho}\right. \Delta, \Gamma, \neg F$ either F or $\neg F$ is false and we obtain the claim immediately by induction hypothesis.

Theorem (Semantical cut elimination)

$\left|\frac{\alpha}{\rho}\right. n \in I_F^\xi$ *implies* $\left|\frac{\alpha}{0}\right. n \in I_F^\xi$. *Hence* $|n|_F \leq \alpha$.

Theorem (Collapsing Theorem)

There is a collapsing function $\Psi_\kappa\colon \mathcal{O} \longrightarrow \Omega_\kappa$ such that for any derivation $\left|\frac{\alpha}{\Omega_\kappa}\right. \Delta$ for a finite set Δ of formulas that do not contain negative occurrences of $I_{F,\kappa}$ and no occurrences of $I_{G,\sigma}$ for initial ordinals $\sigma > \kappa$ we get $\left|\frac{\Psi_\kappa(\alpha)}{\Psi_\kappa(\alpha)}\right. \Delta$.

The basic idea. Elimination of the "ideal" Ω_κ–rules.

$$\dfrac{\dfrac{\Big|\frac{\alpha_0}{\Omega_\kappa}\ \Delta, n \in I^{\Omega_\kappa}_{F,\kappa}}{\Big|\frac{\beta}{\Omega_\kappa}\ \Delta, n \in I^{<\Omega_\kappa}_{F,\kappa}} \qquad \dfrac{\cdots\ \Big|\frac{\beta_\xi}{\Omega_\kappa}\ \Gamma, n \notin I^{\xi}_{F,\kappa}\ \cdots}{\Big|\frac{\beta}{\Omega_\kappa}\ \Gamma, n \notin I^{<\Omega_\kappa}_{F,\kappa}}}{\Big|\frac{\gamma}{\Omega_\kappa+1}\ \Delta, \Gamma}$$

Where Δ and Γ must not contain negative occurrences of $I^{\kappa}_{F,\kappa}$ and no occurrences of $I^{\eta}_{G,\sigma}$ for $\sigma > \kappa$.

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$$\dfrac{\dfrac{\left|\frac{\Psi_\kappa(\alpha_0)}{\Psi_\kappa(\alpha_0)}\right. \Delta, n \in I_{F,\kappa}^{\Omega_\kappa}}{\left|\frac{\beta}{\Omega_\kappa}\right. \Delta, n \in I_{F,\kappa}^{<\Omega_\kappa}} \qquad \dfrac{\cdots \left|\frac{\Psi_\kappa(\beta_\xi)}{\Psi_\kappa(\beta_\xi)}\right. \Gamma, n \notin I_{F,\kappa}^{\xi} \cdots}{\left|\frac{\beta}{\Omega_\kappa}\right. \Gamma, n \notin I_{F,\kappa}^{<\Omega_\kappa}}}{\left|\frac{\gamma}{\Omega_\kappa+1}\right. \Delta, \Gamma}$$

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$$\frac{\left|\frac{\Psi_\kappa(\alpha_0)}{\Psi_\kappa(\alpha_0)}\right. \Delta, n \in I_{F,\kappa}^{\Psi_\kappa(\alpha_0)} \qquad \left|\frac{\Psi_\kappa(\beta_{\Psi_\kappa(\alpha_0)})}{\Psi_\kappa(\beta_{\Psi_\kappa(\alpha_0)})}\right. \Gamma, n \notin I_{F,\kappa}^{\Psi_\kappa(\alpha_0)}}{\left|\frac{\Psi_\kappa(\gamma)}{\Psi_\kappa(\gamma)}\right. \Delta, \Gamma}$$

Where Δ and Γ must not contain negative occurrences of $I_{F,\kappa}^{\kappa}$ and no occurrences of $I_{G,\sigma}^{\eta}$ for $\sigma > \kappa$.

The terms and sentences of the language of *ramified set theory* are constructed by the following clauses

- For any ordinal α the constant L_α is a term of stage α
- If $F(x,\vec{y})$ is a formula in the language of set theory and $\vec{t}$ is tuple of terms of stages less than α then $\{x \in \mathsf{L}_\alpha \,|\; F(x,\vec{t}\,)^{\mathsf{L}_\alpha}\}$ is a term of stage α.
- If $F(\vec{y})$ is a formula in the language of set theory and $\vec{t}$ is a tuple of terms then $F(\vec{t})^{\mathsf{L}_\alpha}$ is a sentence of ramified set theory.

Besides the familiar rules for the logical connectives and the cut rule the canonical rules for the extended language are

($\bigvee$) $\vdash^{\alpha_0}_{\rho} \Delta, F(t)$ for some term t of stage less than η implies $\vdash^{\alpha}_{\rho} \Delta, (\exists x \in \mathsf{L}_\eta)F(x)$ for all $\alpha > \alpha_0$.

($\bigwedge$) $\vdash^{\alpha_t}_{\rho} \Delta, F(t)$ and $\alpha_t < \alpha$ for all terms t of stages less than η implies $\vdash^{\alpha}_{\rho} \Delta, (\forall x \in \mathsf{L}_\eta)F(x)$.

Thank you for your attention