From second order Analysis to subsystems of set theory

Dedicated to Gerhard Jäger on the occasion of his 60th birthday

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1 Introduction

It is a real pleasure for me to be invited to a conference in honor of Gerhard Jägers 60th birthday and I want to thank the organizers for this invitation. My congratulations to Gerhard.

On one side it is a good feeling to see our former "young men" now among the senior notabilities of proof theory, on the other side it is also a weird feeling since it brings your own age home to you.

To honor Gerhard Jägers contribution to proof theory I am going to try to give a non technical though very personally biased account of how we got from subsystems of Second Order Analysis to subsystems of set theory. (Slide 1) This is, however, only one aspect of Gerhard's work. But it is the aspect to which I have the closest bonds.

2 Ordinal analysis for predicative systems

Anyone who knows me will know that I will of course talk about ordinal analysis. To distinguish ordinal analysis from Analysis in the sense of second order number theory I will always capitalize Analysis if I mean second order number theory. Though it is probably difficult to capitalize a word in talking. (Slide 2) Here is a list of the topics I am going to mention. To stress the necessities that brought us to change to subsystems of set theory I have put some emphasis on the time before the change.

2.1 Ordinal analysis

To make clear what I am talking about let us resume some of the basic facts of ordinal analysis. (Slide 3) It means the computation of the proof theoretic ordinal of a mathematical theory. But ordinal analysis is in fact much more than just knowing the proof theoretic ordinal of a theory T. I claim that you know nearly everything about a mathematical theory once you have an ordinal analysis of it. I will, however, not deepen this claim today. Later I will mention an example.

Determining the proof theoretic ordinal of a theory T of course requires that we can talk about well–foundedness in the language of T. Since well–foundedness in an arithmetical language is a genuine Π^1_1–notion this needs a second order language. The situation is, however, not so bad since we can express second order Π^1_1–statements in a first order logic with free second order variables.

(Slide 4) There is a method that goes back to Gerhard Gentzen how such an information can be achieved. We may define the *truth complexity of a* Π^1_1–*sentence* — as I call it — using the ω–completeness theorem as shown on the slide. This form of the ω–completeness theorem is a variant of the Henking–Orey theorem, that either can be obtained from the original theorem by cut–elimination or — more directly — by the use of search trees. The definition of the truth complexity as the shortest cut free ω–proof is then obvious.

(Slide 5) The main theorem which goes back to Gentzen's 1943 paper and has later been improved by Arnold Beckmann is the boundedness theorem that links the order–type of a well–ordering to the truth complexity of the Π^1_1–sentence that expresses its well-foundedness. Therefore it suffices to gauge the truth complexities of the provable formulas of a theory to obtain upper bounds for its proof theoretic ordinal.

Today I will only mention how upper bounds for proof theoretic ordinals can be obtained because there are pretty uniforms methods. Obtaining lower bounds depends more heavily on the peculiarities of the analyzed axiom system.

To obtain upper bounds for proof theoretical ordinals we may proceed in two steps. (Slide 6.) First we embed a formal proof into ω–logic. (2nd click) Then we eliminate cuts and obtain the upper bound by the Boundedness Theorem. (3rd click) For predicative systems the function needed there is essentially the Veblen function. (4th click)

2.2 Ramified Analysis

This works pretty uniform for predicative axiom systems, where I understand "predicative" in a very technical sense which I will explain in a moment. Examples for predicative systems are systems of ramified Analysis (Slide 7) which avoid circular definitions by introducing ramified comprehensions. There is a canonical infinitary proof system for ramified Analysis whose main rules are the two mentioned on the slide.

One good reason to call axiom systems *predicative* if their proof theoretical ordinals are less than or equal to Γ_0 is the famous result by Sol Feferman and Kurt Schütte that fixes the exact bound for autonomously reachable well–orderings. The Schütte–Feferman ordinal Γ_0 (Slide 8) is the least ordinal that is closed under the Veblen function viewed as a binary function. Roughly speaking a well–ordering is autonomously reachable if not only its definition but also the proof of its well–foundedness uses only previously provided means. In terms of ramified Analysis this means that only stages below its own order–type are allowed in its the well–foundedness proof.

However, the methods of predicative proof theory are not restricted to systems with ordinals less than or equal to Γ_0 as Gerhard and his school have shown in their project of metapredicativity. So I would like to draw a (technical) bound between predicative and impredicative systems there, where the methods of predicative proof theory fail.

3 Ordinal analyzes for impredicative axiom systems

Having learned many facts about predicative proof theory in Schütte's lectures and seminars, my interest turned to impredicative axiom systems. The most famous analysis of an impredicative axiom system which existed at that time was that by Gaisi Takeuti [16] for second order number theory with the Π^1_1–comprehension scheme and Bar induction. Yet it was not really an ordinal analysis but rather a consistency proof in the style of Gentzen. In my dissertation I analyzed Takeuti's proof and converted it into an ordinal analysis in terms of an ordinal notation system Σ developed by Schütte. Although I was able to master the technique I did, at that time, not really understand what was going on in Takeuti's reduction procedure. Only much later that became clear by studies of Wilfried Buchholz.

3.1 ν–fold iterated inductive definitions

However, Takeuti's techniques turned out to be very useful in confirming the long conjectured proof theoretic ordinals of axiom systems for iterated inductive definitions – which constitute a perfect sample for impredicative theories. (Slide 9) Here are their essential axioms saying that $I_{F,\sigma}$ is $F(X, I_{\prec\sigma}, x)$–closed and the least such closed class.

The ordinal analysis of the theories ID_ν were first obtained by embedding these theories into systems of iterated Π^1_1–comprehensions which then could be handled by Takeuti's technique.

Although this yielded a correct computation of the upper bounds for the proof theoretic ordinals of the theories ID_ν the method was, due to the complicated reduction procedure a là Takeuti, completely opaque. It was Sol Feferman's constant nagging for a more perspicuous method that kept us (if I may speak also in the name of Wilfried Buchholz) working on the problem. Wilfried Buchholz succeeded in developing his Ω–rules which, however, did not completely satisfy myself.

3.2 A remark on Hilbert's programme

To explain why, I have to give a brief avowal of my motivations for doing ordinal analysis. My starting point is a certain aspect of Hilbert's Programme.

Though I believe that — due to Gödel's second incompleteness theorem — Hilbert's programme failed in so far, that elementary consistency proofs of Analysis are impossible, I nevertheless think that there is another important aspect of Hilbert's programme: The elimination of "ideal objects".

As I see it it is not completely clear what Hilbert understood by "ideal objects" in general. However, there are pretty concrete hints what he meant by "real statements" in contrast to ideal ones. Let me cite a passage of his 1927 talk given in Hamburg. Here is the original citation (Slide 10) but I will not read the text in German but turn directly to my translation. (Slide 11)
But what are the mathematical analogs of experimentally checkable statements? Of course we cannot make experiments in mathematics but we can compute. A good analog for an experimentally checkable statement is therefore a statement that is checkable by a computation, i.e., a Π^0_2–statement. That of course does not mean that we can *prove* Π^0_2–statements by computations but that we can check its instances. A situation comparable to that in physics where we also cannot "prove" the consequences of a theory experimentally but can check instances of

its predictions. (Slide 12)

The analog situation for checking the instances of Π^0_2–consequences of a mathematical theory T could therefore consist in finding a function F_T that helps us to design the "experiment" for the theory T. That means that whenever we have a Π^0_2–sentence $(\forall x)(\exists y)R(x,y)$ we can compute $F_T(m)$ for m a natural number and this number fixes the frame for finitely many "experiments" in which we check $R(m,n)$ for all $n < F_T(m)$ whether $(\exists y)R(m,y)$ is a consequence of T.

3.3 Π^0_2–analysis

Pursuing Hilbert's programme for the elimination of ideal objects then means that the elimination of "ideal means" in a T–proof of a Π^0_2–sentence should provide us with a function F_T that designs an experiment for T. (Slide 13) Clearly the function F_T should be obtainable without reference to ideal means, which excludes functional interpretations with functionals of higher types — of course such interpretations are of interest of their own.

By "ideal methods" I understand axioms or rules that axiomatize ideal objects, e.g., sets obtained by comprehensions, ordinals obtained by reflections, etc. Let us call this program the Π^0_2–analysis of the theory T.

This is an ambitious aim and it was not at all clear from the beginning how far this program could be realized. From our present knowledge we have to admit, that the elimination of ideal methods costs the price of long transfinite recursions. So we should extend Kronecker's aphorism *"the natural numbers a made by God all other is man–made"* to *"the ordinals are made by God all other is man–made"*. Whereas I think that the question in how far the ordinals are God–made and how these ordinals can be represented is one of the deepest problems in foundational research even outside ordinal analysis or even proof theory. But this is a discussion I do not want to enter in this talk.

Π^0_2–analysis is to be seen in contrast to ordinal analysis which we may also call Π^1_1–analysis due to its closeness to the computation of truth complexities of Π^1_1–sentences. However, Π^1_1–analysis can also be considered as a first and less ambitious step towards a Π^0_2–analysis. This is because Π^1_1–statements correspond to Σ_1–statements over $\mathsf{L}_{\omega_1^{CK}}$ with parameters. Π^1_1–statements can thus be viewed as abstractions of Σ^0_1–sentences with parameters in so far that computability, i.e., ω–computability, is replaced by ω_1^{CK}–computability and ω_1^{CK}–computability is easier to handle since there are many ordinals with good closure properties below ω_1^{CK} which is not true for ω.

Admittedly then one of the more practical reasons for looking at Π^1_1–statements was the work of Gentzen (e.g, in his 1943 paper [6]) and his descendants which showed that elimination procedures for proofs of Π^1_1–sentences are feasible.

3.4 A brief résumé of inductive definitions

Coming back to inductive definitions we have fixed–points of positive operators as the analogs of Π^1_1–sentences. — (Slide 13) — The fixed–point of an inductive definitions can be obtained in stages — (Slide 14) — that are defined recursively as shown on the slide.

3.5 Inductive definitions and well–orderings

The stages and the closure ordinal of inductive definitions are closely connected to order–types of well–orderings — (Slide 15) — and we therefore can redefine the proof–theoretic ordinal of a theory in the language of inductive definitions by the stages of elements that provably belong to a fixed–point. Observe that this definition does not need free second order variables in the language of T. This will later be of importance.

3.6 Infinitary logic for inductive definitions

Buchholz' Ω–rule, which I will not write down here, can be regarded as an introduction of higher constructive derivation classes analogous to the introduction of higher number classes in Kleene's $\mathcal{O}$ and its iterations. It therefore corresponds to a hyperjump rule and is thus not directly formalizable within a simple basic theory, say PRA, plus a certain amount of transfinite induction along simple predicates. It did therefore not really match the aims I had in mind. However, I want to emphasize that it computed the correct upper bounds and emerged to be very useful in many other aspects which I cannot touch in this talk.

Nevertheless, I tried to find an alternative method which was closer to the proven techniques of predicative proof theory. The starting point for the development of this alternative technique was the already mentioned fact that the fixed–points of inductive definitions come in stages. The idea was then to develop an infinitary system similar to that known for ramified analysis which had been so successful in determining the bound Γ_0 for predicative Analysis. (Slide 16) Defining the stages of an inductive definition by infinitely long formulas canonically induces an infinitary proof system.

(Slide 16 2nd click) The additional rules for the initial ordinals Ω_κ are then examples of "ideal rules" because they axiomatize the properties of the ideal element Ω_κ.

3.7 Semantical cut–elimination

However, the situation for impredicative axiom systems in general differs essentially from the predicative case.

This again can be well illustrated on the example of inductive definitions.

An essential property of the infinitary calculus for inductive definition is its Boundedness Property —(Slide 17) —, which is due to the fact that all you can prove in α–many steps already holds at stage α.

We have just seen that we only have to deal with sentences to obtain upper bounds for the proof theoretic ordinals for axiom systems in the language of inductive definitions. No free second order variables are required.

However, for theories that only talk about sentences we get cut–elimination nearly for free.

(Slide 18) By a simple semantical lemma we immediately obtain full cut elimination with still keeping control over the lengths of the arising cut free derivations. It is obvious that such a lemma fails in the presence of free second order variables.

Therefore cut elimination alone cannot longer be the hallmark for the ordinal analysis of impredicative axiom systems as it is the case for predicative ones.

That semantical cut elimination does not make ordinal analysis trivial is caused by the rules for the initial ordinals Ω_κ which are supposed to be ordinals bigger than or equal to ω_1^{CK}. They therefore produce derivations of length bigger than ω_1^{CK}.

However, it is easy to see that the proof theoretical ordinal of any axiom system is always an ordinal less than ω_1^{CK}. Therefore the semantical cut elimination theorem is of no use in ordinal analysis since the resulting ordinals are in general too big. This also shows that cut elimination alone cannot suffice for an ordinal analysis.

3.8 Local predicativity

The new feature that is needed is "collapsing", — (Slide 19) – a procedure that collapses derivations of formulas that only contain positive occurrences of Ω_κ fixed–points into derivations of lengths below Ω_κ. Intuitively it is clear that such a derivation should exists since no Ω_κ–branching inferences are needed to derive formulas that do not contain negative occurrences of a Ω_κ–fixed–point. The problem is to find a procedure that transfers the original derivation and a function that computes $\Psi_\kappa(\alpha)$ from the data κ and α.

Since ordinals are hereditarily transitive they are not collapsible. Therefore the collapsing function must not be defined on a segment of the ordinals. The class $\mathcal{O}$ has to be a proper subclass of the ordinals with sufficiently many gaps.

Finding the right subclass is by far the most difficult problem for stronger systems (such as full reflection and stability). I will, however, not go into more details here.

The idea was then to combine collapsing with boundedness to obtain an ordinal analysis of iterated inductive definitions. Although cut elimination is not longer the hallmark of impredicative proof theory the main problem is still caused by cuts. To use boundedness we need to know that there are no negative occurrences of Ω_κ–fixed–points in the whole derivation. Therefore we have to eliminate all cuts with complexities above Ω_κ. The basic idea to do that is, however, so simple that we can easily explain it on ”one” slide. (Slide 20 4clicks)

Clearly, the technical details are much more complex but it worked perfectly for iterated inductive definitions although the definition of the collapsing functions at that time were still pretty clumsy.

4 Towards set theory

Of course is was tempting to try to transfer this technique to ramified analysis in order to obtain direct analyzes for subsystem of classical Analysis. This, however, met unexpected difficulties. It was of course not too difficult to transfer the technique of local predicativity to the axiom system of parameter free Π^1_1–comprehension — this is in principle the same system as ID_1 — but already Π^1_1–comprehension with parameters — which corresponds to the theory $\mathsf{ID}_{<\omega}$ — caused difficulties. Difficulties that appeared not to be insurmountable but needed so much coding work that it became practically unmanageable.

One point was that there appeared sets that were not longer sets of natural numbers and thus not longer sets in the ramified analytical hierarchy. They needed to be coded into sets of natural numbers.

The idea was therefore: why not work directly with arbitrary sets and not merely with sets in the ramified analytical hierarchy? Since the ramified analytical hierarchy is Gödel’s constructible hierarchy intersected with the power set of the natural numbers it was obvious to try to work in the constructible hierarchy itself.

Searching the literature we found an article by Sol Feferman [4] in which he treated "predicatively reducible systems of set theory" and a paper by Harvey Friedman [5] treating "set theoretic foundations for constructive analysis". These papers, however, reduced set theoretic axiom systems to known subsystems of Analysis. What we wanted to do was the converse way.

4.1 Ramified set theory

The first aim was to design a language for "ramified set theory". That was not too difficult. One could more or less directly use the language of the constructible hierarchy with its stages L_α as additional constants. (Slide 22)

Having designed the language there are again canonical rules for an infinitary proof system for this language. This is a bit oversimplified but the details do not matter here. It already looks very similar to predicative Analysis and we are not longer forced to code sets into sets of natural numbers. Nevertheless, there were many difficulties to overcome.

Fortunately at that time there was a clever student in Munich, Gerhard Jäger, with whom we could discuss the situation intensively and who was looking for a diploma thesis. I encouraged Schütte — since I had not yet passed my "habilitation" I was not allowed to supervise diploma students myself — to let Gerhard work on this problem. This was very ambitious for a diploma thesis because there were hardly patterns for doing proof theory directly in the constructible hierarchy. As a matter of course, Gerhard mastered the problem, starting with a still predicative system. This led to an excellent diploma thesis partly published as '*"Beweistheorie von KPN"*[8]. Here I should perhaps also mention the big influence of Barwise's book [1] on *"admissible sets and structures"* on our discussions. An influence similar to that of Moschovakis' book *"Elementary induction on abstract structures"* [12] on our work on inductive definitions.

Gerhard received his diploma degree with distinction and continued to work in this direction. In his dissertation *"Die konstruktible Hierarchie als Hilfsmittel zur beweistheoretischen Untersuchung von Teilsystemen der Analysis"* [7] he extended this work further and it culminated in his Habilitationsschrift *"Theories for admissible sets. A unifying approach"*[10] published by Bibiliopolis in 1986.

He so laid the fundament for all further research in this direction.

Already in 1982 we had a joint paper [11] published in the "Sitzungsberichte der Bayerischen Akademie der Wissenschaften", unfortunately in German, in which we gave an ordinal analysis of Δ^1_2–comprehension with the (classical) bar induction via an ordinal analysis of the theory KPi, a set theory that axiomatizes an admissible universe which is also the union of admissible universes.

I mention this result since it may serve as an example for my previous remark that *"you know nearly everything about a theory once you have an ordinal analysis of it"*. Having analyzed KPi Gerhard Jäger succeeded in proving the open conjecture that Feferman's theory T_0 for explicit mathematics is equivalent to Δ^1_2 comprehension with bar induction (cf.[9]). A claim which then seemed to be impenetrable by other means. He solved it by giving a well–ordering proof for the ordinal notation system obtained by the analysis of KPi within the theory T_0. There are of course also other examples of *"knowing nearly everything"*, mostly connected with Π^0_2 analyzes, which I cannot go into further.

4.2 More recent developments

Since then many advances took place. Wilfried Buchholz [3] introduced operator controlled derivations as a simplification of local predicativity. As a matter of fact this is much more than just a simplification. Local predicativity fails for theories stronger than Π_2–reflection. In his analysis of Π_3–reflection Michael Rathjen [13] introduced a new technique based on thinning operations on the ordinals. A technique which led to analyzes of theories up to the strength of Σ_1–Separation, a theory that is equivalent to Π^1_2–comprehension [14]. Operator controlled derivations play an important role in this analyzes.

There is also progress in extending the elimination procedures to proofs of Π^0_2–statements — which were my original aim.

The basis therefor was laid by Andreas Weiermann. There are two papers, one joint with Adam Cichon and Wilfried Buchholz in which they developed a new approach to subrecursive hierarchies which is essential for such analyzes, the other, joint with Benjamin Blankertz, in which they used such hierarchies to obtain Π^0_2–analyzes. Benjamin Blankertz later developed the technical details in a very general setting in his dissertation [2]. Jan Carl Stegert [15] in his dissertation simplified Blankertz' work and extended it to axiom systems for reflection and stability.

I myself am busy to collect all these results in a monograph about the proof theory of stability. This is work in progress.

References

[1] J. BARWISE, **Admissible sets and structures**, Perspectives in Mathematical Logic, Springer-Verlag, Berlin/Heidelberg/New York, 1975.

[2] B. BLANKERTZ, **Beweistheoretischen Techniken zur Bestimmung von Π^0_2–Skolem Funktionen**, Dissertation, Westfälische Wilhelms-Universität, Münster, 1997.

[3] W. BUCHHOLZ, *A simplified version of local predicativity*, **Proof theory** (P. Aczel et al., editors), Cambridge University Press, Cambridge, 1992, pp. 115–147.

[4] S. FEFERMAN, *Predicatively reducible systems of set theory*, **Axiomatic set theory** (D. S. Scott and T. J. Jech, editors), vol. 2, Proceedings of Symposia in Pure Mathematics, vol. 13, American Mathematical Society, Providence, 1974, pp. 11–32.

[5] H. M. FRIEDMAN, *Set theoretic foundations for constructive analysis*, **Annals of Mathematics**, vol. 105 (1977), pp. 1–28.

[6] G. GENTZEN, *Beweisbarkeit und Unbeweisbarkeit von Anfangsfällen der transfiniten Induktion in der reinen Zahlentheorie*, **Mathematische Annalen**, vol. 119 (1943), pp. 140–161.

[7] G. JÄGER, **Die konstruktible Hierarchie als Hilfsmittel zur beweistheoretischen Untersuchung von Teilsystemen der Mengenlehre und Analysis**, Dissertation, Ludwig-Maximilians-Universität, Munich, 1979.

[8] ——, *Beweistheorie von KPN*, **Archiv für Mathematische Logik und Grundlagenforschung**, vol. 20 (1980), pp. 53–63.

[9] ——, *A well ordering proof for Feferman's theory T_0*, **Archiv für Mathematische Logik und Grundlagenforschung**, vol. 23 (1983), pp. 65–77.

[10] ——, **Theories for admissible sets. A unifying approach to proof theory**, Studies in Proof Theory, Lecture Notes, vol. 2, Bibliopolis, Naples, 1986.

[11] G. JÄGER AND W. POHLERS, *Eine beweistheoretische Untersuchung von $(\Delta^1_2\text{-}CA) + (BI)$ und verwandter Systeme*, **Bayerische Akademie der Wissenschaften, Sitzungsberichte 1982**, (1983), pp. 1–28.

[12] Y. N. MOSCHOVAKIS, **Elementary induction on abstract structures**, Studies in Logic and the Foundations of Mathematics, vol. 77, North-Holland Publishing Company, Amsterdam, 1974.

[13] M. RATHJEN, **Eine Ordinalzahlanalyse der $\mathbf{\Pi_3}$-Reflexion**, Habilitationsschrift, Westfälische Wilhelms-Universität, Münster, 1992.

[14] ——, *An ordinal analyis of parameter free Π^1_2-comprehension*, **Archive for Mathematical Logic**, vol. 48/3 (2005), pp. 263–362.

[15] J.-C. STEGERT, **Ordinal proof theory of Kripke–Platek set theory augmented by strong reflection principles**, Ph.D. Thesis, Westfälische Wilhelms-Universität, Münster, 2011.

[16] G. TAKEUTI, *Consistency proofs of subsystems of classical analysis*, **Annals of Mathematics**, vol. 86 (1967), pp. 299–348.